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Optimal design of small ΔT thermoelectric generation systems

James W. Stevens *

Department of Mechanical Engineering, Mississippi State University, P.O. Box ME, Mississippi State, MS 39762, USA

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Abstract

Thermoelectric generation systems for utilizing waste heat and ambient temperature swings have been proposed that would involve temperature differences on the order of 1–10 K. The inherently low thermodynamic efficiency of such systems requires that a relatively large amount of heat pass through the system for a given rate of electricity production. The coupled design of the thermoelectric module and the hot side and cold side heat exchangers is fundamental to the creation of a useful device. An approximate optimal design is derived that is applicable to systems with a small temperature difference between the reservoirs. For a fixed thermal resistance in the heat exchangers, the optimal configuration splits the total temperature drop evenly between the thermoelectric module and the heat exchangers. For heat exchanger thermal resistances that vary with time, optimal configuration are derived for linear, sawtooth function and sinusoidal variations. The application of these equations to a typical system design is described.

Keywords: Thermoelectric generation; Waste heat utilization; Optimal design

1. Introduction

Thermoelectric devices are used in a wide variety of applications for portable or remote power generation and heat pumping. Inasmuch as thermoelectric generators function as heat engines, they are subject to the constraints of the second law of thermodynamics on the maximum efficiency. This limit may be expressed by $\eta_C = \Delta T / T_H$. Thus, a principal focus for improving generator efficiency has been in seeking to increase the temperature range through which

* Tel.: +1-662-325-8231; fax: +1-662-325-7223.
E-mail address: stevens@me.msstate.edu (J.W. Stevens).

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Nomenclature

α	Seebeck coefficient for a junction
γ_n, γ_p	thermoelectric leg aspect ratio (A_n/l or A_p/l)
η_C	Carnot efficiency
η_t	thermal efficiency
η_{TE}	thermal efficiency of thermoelectric module
ρ_n, ρ_p	electrical resistivity of each thermoelectric material
Σ	sum of external thermal resistances: $R_H + R_C$
A	face area of thermoelectric module
A_n, A_p	thermoelectric material leg cross-sectional area
f	ratio of thermoelectric material cross-section to total face area ($A/(A_n + A_p)$)
I	current
K	overall thermal conductance
k_n, k_p	thermal conductivity of each thermoelectric material
l	thermoelectric material leg length
Q	heat flow
R_H, R_C	thermal resistances of hot side and cold side heat exchangers
R_e	internal electrical resistance
$R_{e,o}$	external electrical resistance
R_i	thermal resistance internal to thermoelectric module
T_H	hot reservoir temperature
T_C	cold reservoir temperature
T_1	junction temperature on hot side of thermoelectric module
ΔT_{TE}	temperature difference across thermoelectric module
ΔT_{HC}	temperature difference between hot and cold reservoirs
U_A, U_G	overall heat transfer coefficients of air side and ground side heat exchangers
$\dot{W}$	power output

thermoelectric materials will function. Another potential use of thermoelectric generators is in the conversion of low grade (low temperature) heat into electricity. One proposal would generate power from the difference between the fluctuating daily air temperature and the relatively constant ground temperature. In this case and many other waste heat applications, temperature differences would be on the order of 1–10 K. Since these systems are constrained to low thermodynamic efficiency, a comparatively large amount of heat must be moved in order to generate a given quantity of electricity. Thus, the design of the hot side and cold side heat exchangers as well as the matching of the heat exchanger and thermoelectric designs assume important roles in the creation of a useful device. Fortunately, the very small power production relative to the total heat flow enables approximations to be used in the thermoelectric module design that include the heat exchangers in a simple way. This paper describes and illustrates an approximate procedure for the

optimal coupled design of a thermoelectric generator and the accompanying heat exchangers for small ΔT applications.

2. Relevant prior work

There is an abundant body of literature on thermoelectric generators and applications of thermoelectric generators, but the vast majority of this literature focuses on temperature differences one to two orders of magnitude greater than those relevant to this work. Nevertheless, several applications of thermoelectric generators have been explored for circumstances somewhat similar to those considered here.

Lemley [1] presented an unusual application of a thermoelectric generator designed to produce power from the long wave infrared radiation leaving the surface of the earth. This project involved generating power for a high altitude, long duration communications platform. Energy would be collected by radiation from the earth's surface and rejected by radiation into space. A thin film thermoelectric device configuration was devised and tested. A total temperature difference of 58°C was used in the design.

Benson and Jayadev [2] presented the idea of using thermoelectric generators in very large scale installations to produce useful amounts of electricity from low grade heat sources. One possibility was to use thermoelectric generators in an ocean thermal energy conversion (OTEC) system. The proposed temperature difference for that case was 5–25°C. It was shown that some commercially available thermoelectric materials have an acceptable figure to merit in this temperature range. It was assumed that a performance of 20% of the Carnot efficiency could be achieved from the thermoelectric devices. Other potential sources of energy that were discussed included geothermally heated ocean water (85°C), solar ponds (50°C), natural lake thermoclines (10–20°C) and utility power plant waste heat (15°C). Long term capital costs were taken into account in the discussion.

Benson and Tracy [3] discussed the development of thin film thermoelectric generators for use with low grade thermal energy applications such as those discussed in Benson and Jayadev [2].

Chen [4] performed the thermodynamic analysis of a thermoelectric generator powered by direct solar radiation. Four classes of important irreversibilities were identified, and their effects were included in the analysis. These irreversibilities included: finite rate heat transfer between the device and the surroundings, heat leaks internal to the device, Ohmic heat production and heat losses in the solar collector. The main conclusions of the analysis established the performance limits for solar powered thermoelectric devices.

Wu [5] analyzed the thermodynamics of a thermoelectric generator that would operate from industrial waste heat. A very similar analysis was performed (Wu [6]) for a generator operating from a solar pond. In both cases, the thermodynamic efficiency and specific work output were emphasized, and comparisons with theoretically obtainable efficiencies were presented.

Henderson [7] developed an analytical model for a thermoelectric generator with small ΔT that included the hot and cold side heat exchangers. The overall system power was optimized using Lagrange multipliers on a system of four non-linear equations. The solution could be found via an iterative approach, but no closed form solution was determined. Analyses of several complete

systems led to an empirical conclusion that the total temperature difference should be split equally between the thermoelectric module and the external heat exchangers.

3. Analysis

It can be shown (e.g. Ref. [8]) that starting from an expression for the thermal efficiency of a thermoelectric module:

$$\eta_t = \frac{I^2 R_{e,o}}{K\,\Delta T_{TE} + \alpha T_H I - \frac{1}{2} I^2 R_e} \tag{1}$$

the maximum thermal efficiency will be obtained if

$$\frac{\gamma_n}{\gamma_p} = \left(\frac{\rho_n k_p}{\rho_p k_n}\right)^{1/2} \tag{2}$$

and for maximum power:

$$\frac{\gamma_n}{\gamma_p} = \left(\frac{\rho_n}{\rho_p}\right)^{1/2} \tag{3}$$

In both cases, the device is optimized with respect to fixed hot and cold side temperature at the ends of the thermoelectric elements. Also in both cases, the optimization of the thermoelectric module specifies only the *ratio* of the aspect ratios. In application, the temperatures at the ends of the thermoelectric elements would not be fixed unless the elements were in infinitely good thermal contact with the hot and cold thermal reservoirs. Instead, hot side and cold side heat exchangers with some overall thermal resistance will be interposed between the thermal reservoir and the thermoelectric elements. The inclusion of these thermal resistances to complete the thermoelectric system design is described below.

3.1. Optimal configuration

A simple thermal resistance model of the thermoelectric generator system is shown in Fig. 1. In this model, heat flows from a high temperature reservoir at T_H, through a heat exchanger with overall thermal resistance R_H, through a thermoelectric module with thermal resistance R_i and, finally, through another heat exchanger with overall thermal resistance R_C to the cold reservoir temperature at T_C. For the case of small ΔT, the amount of energy taken away as electricity may be neglected, since it will be very small relative to the total energy flow. Thus, $Q_H \approx Q_L \equiv Q$. It is possible to determine the device output as a function of the temperature difference between the hot reservoir and the cold reservoir. This will include within it the temperature drops through the two heat exchangers. The thermal resistances for each heat exchanger account for all temperature change between the reservoir temperature (T_H or T_C) and the junctions of the thermoelectric module. Starting with definitions for the thermal efficiency of the thermoelectric module and the Carnot efficiency for the module:

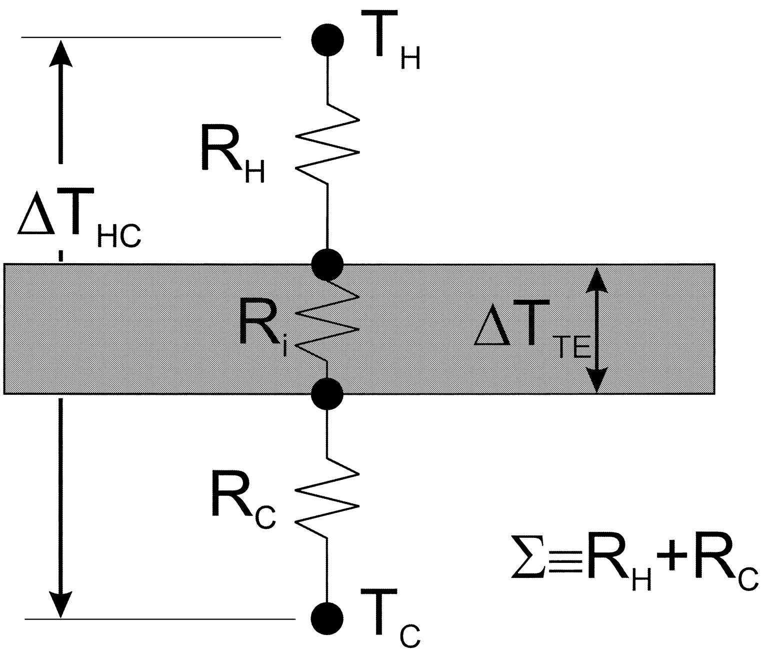

Fig. 1. Thermal circuit model.

$$\eta_{TE} = \frac{\dot{W}}{Q} \tag{4}$$

$$\eta_{C} = \frac{\Delta T_{TE}}{T_1} \tag{5}$$

a second law efficiency can be defined as the ratio of the module efficiency to its Carnot efficiency:

$$\eta_2 = \frac{\eta_{TE}}{\eta_C} = \frac{\dot{W} T_1}{Q \Delta T_{TE}} \tag{6}$$

It can be shown that to a first order approximation, this ratio is constant over a small temperature range. Eq. (6) can be rearranged:

$$\Delta T_{TE} = \frac{\dot{W} T_1}{Q \eta_2} \tag{7}$$

Also, from the definition of thermal resistance:

$$Q = \frac{\Delta T_{TE}}{R_i} \tag{8}$$

and

$$Q = \frac{\Delta T_{HC}}{R_i + \Sigma} \tag{9}$$

where $\Sigma \equiv R_H + R_C$. That is, Σ accounts for all thermal resistance (hot side and cold side) external to the thermoelectric module, while R_i represents the thermal resistance internal to the thermoelectric module. Finally, combining Eqs. (7)–(9) to eliminate ΔT_{TE} and Q:

$$\dot{W} = (\Delta T_{HC})^2 \frac{R_i}{(R_i + \Sigma)^2} \frac{\eta_2}{T} \tag{10}$$

For small ΔT_{HC}, $T_1 \approx T \equiv (T_H + T_C)/2$. This expression gives the power output of the thermoelectric module in terms of the hot and cold reservoir temperatures, the thermal resistances and the second law efficiency. It shows that the power output is proportional to the second law efficiency, inversely proportional to the absolute temperature and proportional to the square of the reservoir temperature difference. Only the sum of the external thermal resistances (hot side and cold side) is important, not their individual sizes.

Further illustration of the effects of thermal resistances is contained in Figs. 2 and 3. Fig. 2 shows the power output as a function of external thermal resistance if everything else is held constant. As might be expected, the power output increases monotonically with decreasing external thermal resistance. With no external thermal resistance, the device operates as it would with both surfaces of the thermoelectric module held at the reservoir temperatures.

Fig. 3 shows the variation in power as a function of R_i with everything else held constant. In this case, there is a maximum in the power generation for the complete system. This maximum occurs because at high R_i, too little heat can flow through the system, while at low R_i, there is only a small temperature drop across the thermoelectric module. By taking the derivative of power with respect to R_i and setting it equal to zero, it can be shown that the maximum in power occurs when $R_i = \Sigma$:

$$\frac{\partial \dot{W}}{\partial R_i} = \left(\frac{1}{(R_i + \Sigma)^2} - \frac{2R_i}{(R_i + \Sigma)^3} \right) \frac{\Delta T_{HC}^2 \eta_2}{T} \tag{11}$$

This matches the conclusion reached empirically by Henderson [7] after repeated analyses with a more exact and more complex optimization and numerical evaluation procedure.

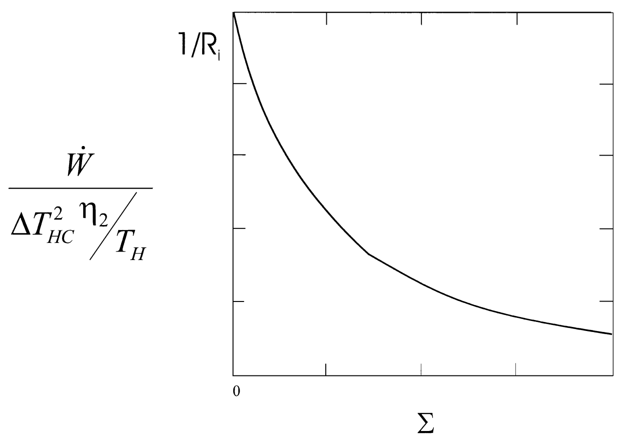

Fig. 2. Variation in power output for constant internal thermal resistance.

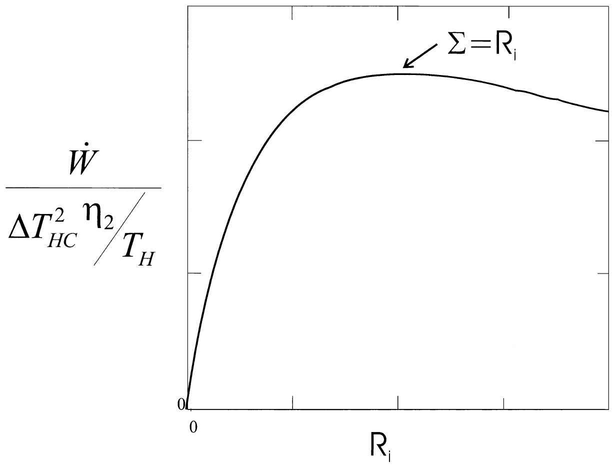

Fig. 3. Variation in power output for constant external thermal resistance.

3.2. Variable Σ

In application, the values of R_H and R_C could vary with changing conditions. For example, for a system operating from ambient temperature differences, this could occur because of changes in natural convection and radiation on the heat exchanger interface with the environment. Since R_i will be fixed once the thermoelectric module is constructed, it will be impossible to maintain $R_i = \Sigma$ when Σ varies. In many cases, changes in Σ could be represented as function of time: $\Sigma(t)$. Therefore, it is desirable to find the value of R_i that will maximize the average power over some fixed time period. The average power over such a period is:

$$\overline{\dot{W}} = \frac{1}{t_2 - t_1} \int_{t_1}^{t_2} \dot{W}(\Sigma(t), R_i)\,\mathrm{d}t \tag{12}$$

where $\dot{W}$ comes from Eq. (10), and Σ is a function of time, $\Sigma(t)$. A simple case to analyze is a general linear variation represented by:

$$\Sigma(t) = \sigma t \tag{13}$$

In Eq. (13), σ is a constant. Combining Eqs. (10), (12) and (13) yields:

$$\overline{\dot{W}} = \frac{1}{t_2 - t_1} \int_{t_1}^{t_2} (\Delta T_{HC})^2 \frac{\eta_2}{T} \frac{R_i}{(\sigma t + R_i)^2}\,\mathrm{d}t \tag{14}$$

As a first approximation, it is assumed that all quantities in Eq. (14) are independent of time except, of course, t itself. The weakest aspect of this assumption is the inclusion of ΔT_{HC}, since it would sometimes be changes in temperature that would drive the changes in thermal resistance. Changes in ambient temperature would result in relatively smaller changes in ΔT_{HC} than in the temperature differences that affect the thermal resistance simply because the latter are much smaller (on the order of one quarter) than the total difference. Nevertheless, the effect of the dependence of

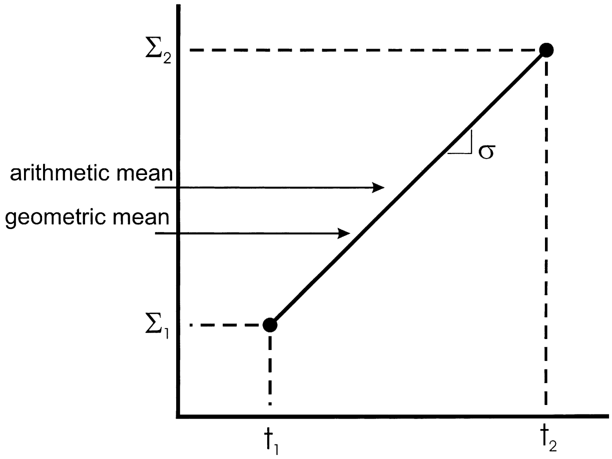

Fig. 4. General linear variation in $\Sigma(t)$.

ΔT_{HC} on time may not always be negligible. However, as a first approximation, it will be neglected in this development. After performing the integration indicated in Eq. (14), the result is:

$$\overline{\dot{W}} = (\Delta T_{AG})^2 \frac{\eta_2}{T} \frac{R_i}{(\sigma t + R_i)^2} \frac{1}{t_2 - t_1} \left(\frac{R_i}{\sigma} \left(\frac{1}{\sigma t_1 + R_i} \right) - \left(\frac{1}{\sigma t_2 + R_i} \right) \right) \tag{15}$$

In order to determine the value of R_i that will yield the maximum average power over the interval t_1 to t_2, the derivative of Eq. (15) with respect to R_i is set equal to zero resulting in:

$$R_i = (\sigma^2 t_2 t_1)^{1/2} = \sqrt{\Sigma_2 \Sigma_1} \tag{16}$$

Eq. (16) indicates that the appropriate value for the internal thermal resistance is the geometric mean of the two values of external thermal resistance at the endpoints. The arithmetic and geometric means are illustrated in Fig. 4. The geometric mean is always lower than the arithmetic mean, and the difference becomes more pronounced with increasing difference between the two input numbers.

3.3. Other unsteady variations in Σ

In light of the daily variations in conditions that may be applicable to many types of low grade heat source applications, it is also of interest to examine the effect of a sinusoidal variation in thermal resistance. This can be represented generally by the function:

$$\Sigma(t) = \sigma \sin(t) + B \tag{17}$$

In Eq. (17), σ and B are constants, where σ is always less than B since the thermal resistance cannot be less than zero.

Substituting Eq. (17) into Eqs. (10) and (12), a process corresponding to Eqs. (14)–(16) can be performed. While the calculations are more complex, the steps are identical to the simpler linear case outlined above.

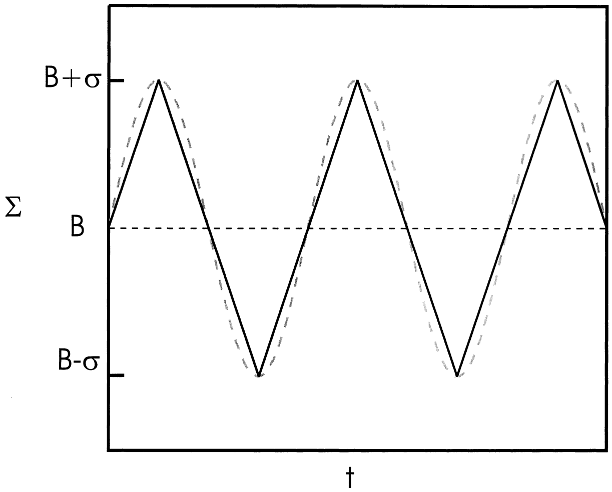

Fig. 5. Sawtooth wave function for $\Sigma(t)$.

Before presenting the results of that analysis, it is useful to apply the result of the linear analysis to obtain a rough approximation to the sinusoidal case. Fig. 5 shows a sawtooth wave function with the same amplitude and frequency as the general sine wave of Eq. (17). Since the sawtooth wave is composed entirely of line segments, all with the same magnitude for endpoints, it can be treated with the results of the linear analysis (Eq. (16)):

$$R_{\mathrm{i}} = \sqrt{(B+\sigma)(B-\sigma)} \tag{18}$$

which can be written as:

$$\frac{R_{\mathrm{i}}}{B} = \left(1 - \left(\frac{\sigma}{B}\right)^2\right)^{1/2} \tag{19}$$

In Eq. (19), the optimum value of the internal thermal resistance is a function of both the average value of the external thermal resistance (B) and the magnitude of the fluctuations (σ). As might be expected, as the fluctuations disappear ($\sigma \Rightarrow 0$) the optimal thermal resistance approaches that of the constant case: $R_{\mathrm{i}} = B$.

The result of the analysis for the sinusoidal case consists of the positive roots of a third order polynomial:

$$\left(\frac{R_{\mathrm{i}}}{B}\right)^3 + \left(\frac{R_{\mathrm{i}}}{B}\right)^2 - \left(1 - 2\left(\frac{\sigma}{B}\right)^2\right)\left(\frac{R_{\mathrm{i}}}{B}\right) - \left(1 - \left(\frac{\sigma}{B}\right)^2\right) = 0 \tag{20}$$

Eq. (19) and the positive roots of Eq. (20) are plotted together in Fig. 6. As expected, the sinusoidal result is similar in shape to the sawtooth wave of the same frequency and amplitude. The sinusoidal result is always less than that of the sawtooth wave. Comparison of the two results gives some feeling for the sensitivity of the optimum to the generating function. This is useful, since practical inputs would rarely follow exact sinusoidal or sawtooth function shapes.

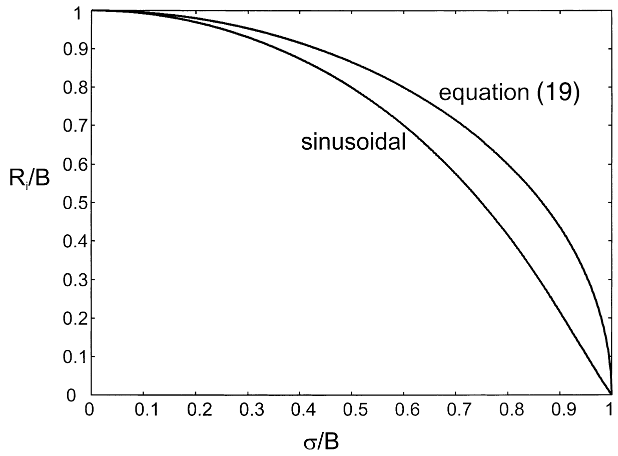

Fig. 6. Results for sinusoidal and sawtooth wave functions.

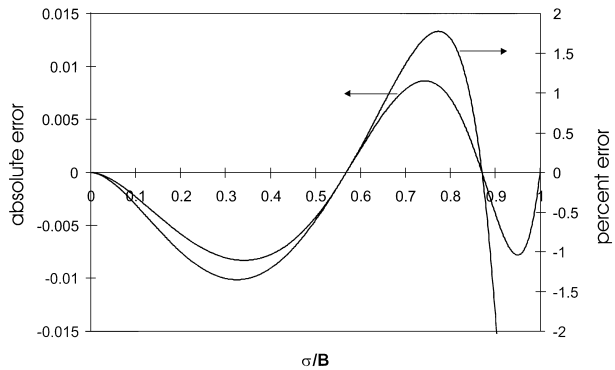

Fig. 7. Error in Eq. (21).

For design purposes, it is desirable to have a closed form solution to Eq. (20). The curve shown in Fig. 6 is very smooth and can be fit to any desired level of accuracy by using a sufficiently high order polynomial. However, a remarkably simple function which provides an excellent fit is:

$$\frac{R_{\mathrm{i}}}{B} = 1 - \left(\frac{\sigma}{B}\right)^{2.35} \tag{21}$$

The absolute and percent error of this function are plotted in Fig. 7. The error is less than two percent for $\sigma/B \leqslant 0.9$.

3.4. Design illustration

The foregoing results can be useful for the overall design of a thermoelectric generation system. This design could take many paths, but for the purposes of illustration, it will be assumed that the reservoir temperatures and material properties are fixed and the total power output is specified.

The external thermal resistance will be taken as a constant. The overall heat transfer coefficients for the heat exchangers are assumed to be calculated from a heat transfer analysis. The leg lengths and aspect ratios for the n and p type materials and the total area of the device need to be determined.

The system design would proceed as follows:

For a specified power requirement, operating condition, and thermoelectric material, the optimum internal and external thermal resistance can be determined from Eq. (10) which is rearranged here as:

$$\Sigma = R_{\mathrm{i}} = \frac{\eta_2 \Delta T_{\mathrm{HC}}^2}{4\dot{W}T} \tag{22}$$

The aspect ratios, γ_{p} and γ_{n}, can then be determined from combining the internal thermal resistance equation:

$$R_{\mathrm{i}} = (\gamma_{\mathrm{n}} k_{\mathrm{n}} + \gamma_{\mathrm{p}} k_{\mathrm{p}})^{-1} \tag{23}$$

with the expression for optimal aspect ratio, Eq. (2) or Eq. (3). For example, combining Eq. (2) with Eq. (23) yields:

$$\gamma_{\mathrm{p}} = \left(R_{\mathrm{i}} \left(k_{\mathrm{n}} \left(\frac{\rho_{\mathrm{n}} k_{\mathrm{p}}}{\rho_{\mathrm{p}} k_{\mathrm{n}}} \right)^{1/2} + k_{\mathrm{p}} \right) \right)^{-1} \tag{24}$$

and

$$\gamma_{\mathrm{n}} = \left(\frac{\rho_{\mathrm{n}} k_{\mathrm{p}}}{\rho_{\mathrm{p}} k_{\mathrm{n}}} \right)^{1/2} \gamma_{\mathrm{p}} \tag{25}$$

For the case of maximum power density, a similar procedure could be used with Eq. (23) and Eq. (3).

From a heat transfer analysis of the air side and ground side heat exchangers, overall heat transfer coefficients, U_{H} and U_{C} can be determined. The required face area of the thermoelectric module can be determined from the definition of thermal resistance:

$$A = \frac{1}{\Sigma} \left(\frac{1}{U_{\mathrm{H}}} + \frac{1}{U_{\mathrm{C}}} \right) \tag{26}$$

Since the thermoelectric material does not completely cover the module faces, the face area is related to thermoelectric material cross-sectional area by a coverage factor, f:

$$A_{\mathrm{n}} + A_{\mathrm{p}} = Af \tag{27}$$

Assuming that all thermoelectric module legs have equal length:

$$l = \frac{Af}{\gamma_{\mathrm{n}} + \gamma_{\mathrm{p}}} \tag{28}$$

and the geometry has been fully specified.

4. Conclusion

Derivations and expressions for the optimal internal thermal resistance of a small ΔT thermoelectric generator with steady and time varying external thermal resistance were presented. The three general time varying cases included a linear variation and sawtooth and sinusoidal variations. For fixed external thermal resistance, the internal thermal resistance should match the external thermal resistance for maximum power production. This is equivalent to saying that the total reservoir temperature difference should be divided evenly between the thermoelectric module and the rest of the system (the two heat exchangers). The linear variation in external thermal resistance resulted in the optimum being at the geometric mean of the two endpoints. The optimum for the sinusoidal variation could be expressed accurately in a closed form function of the mean value and the fluctuation magnitude. A procedure for using the results in a system design was illustrated.

Acknowledgements

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